

Non-adiabatic transitions in a massless scalar field

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Abstract

We consider the dynamics of a massless scalar field with time-dependent sources in the adiabatic limit. This is an example of an adiabatic problem without spectral gap. The main goal of our paper is to illustrate the difference between the error of the adiabatic approximation and the concept of non-adiabatic transitions for gapless systems. In our example the non-adiabatic transitions correspond to emission of free bosons, while the error of the adiabatic approximation is dominated by a velocity-dependent deformation of the ground state of the field. In order to capture these concepts precisely, we show how to construct super-adiabatic approximations for a gapless system.

1 Introduction

The adiabatic theorem of quantum mechanics is usually stated and proved for systems with a finite spectral gap. Mathematically it is known since at least 1998 [9, 3] that in a weaker sense the theorem remains valid even for systems without spectral gap as long as the spectral projections of the time-dependent Hamiltonian are sufficiently regular functions of time, see also [17]. Its validity has been shown even for resonances [1] and for open systems [2]. In recent years the problem of adiabaticity for gapless systems attracted also interest in physics. For example, in [13] the authors consider adiabatic changes in the coupling of a scalar field. In a finite volume the ground state of the field is separated by a gap from the excited states. However, in the thermodynamic limit the gap closes and the spectrum of the field becomes absolutely continuous. The authors find that in the thermodynamic limit the error in the adiabatic approximation depends on the spatial dimension and other parameters controlling the spectral density of the field. They distinguish three regimes: one where the adiabatic theorem holds with the same error estimate as in the gapped case, one where it holds with a different power law and one where it fails completely. In all cases, however, they identify the error of the adiabatic approximation with the size of the non-adiabatic transitions. One main goal of our paper is to explain why this is, in general, not a valid identification and why the concept of super-adiabatic approximations is useful also in the gapless case.

To this end we consider a specific gapless model in the adiabatic limit, namely a massless scalar field with time dependent sources. The goal is to exhibit a number of subtleties in adiabatic theory that are known in the gapped case – but often not expressed sufficiently clearly – also for the case of gapless systems. In particular we emphasize the difference between the error in the adiabatic approximation and what physically should be considered non-adiabatic transitions. To be more specific, let us briefly recall the situation in the presence of a spectral gap. Let $H(t)$ be a time-dependent Hamiltonian and $E(t)$ an eigenvalue with spectral projection $P(t)$. Under appropriate regularity conditions on $H(t)$ and the assumption that $E(t)$ is separated by a gap from the rest of the spectrum of $H(t)$, the adiabatic theorem states that the solution of the Schrödinger equation

$$i\varepsilon \frac{d}{dt} \psi(t) = H(t)\psi(t) \quad \text{with} \quad \psi(0) \in \text{ran } P(0)$$

satisfies

$$\|(1 - P(t))\psi(t)\| = \mathcal{O}(\varepsilon), \tag{1}$$

i.e. that any solution starting in the spectral subspace $\text{ran } P(0)$ evolves into the subspace $\text{ran } P(t)$ up to an error of order ε . This error estimate, which contributes to the error of the adiabatic approximation, is optimal in the sense that generically the piece $(1 - P(t))\psi(t)$ of the solution in the orthogonal complement of $\text{ran } P(t)$ is really of order ε and not smaller for any finite time t where $\frac{d}{dt}H(t) \neq 0$. However, if $H(t)$ is constant outside of a bounded interval, say $[0, T]$, and $N + 1$ -times continuously differentiable, then clearly $\psi(t) \in \text{ran } P(t)$ for $t \leq 0$, but also

$$\|(1 - P(t))\psi(t)\| = \mathcal{O}(\varepsilon^N) \quad \text{for } t \geq T. \quad (2)$$

Thus the non-adiabatic transitions into the orthogonal complement of $\text{ran } P(t)$ after a compactly supported adiabatic change of the Hamiltonian are much smaller than $\mathcal{O}(\varepsilon)$. What happens is that during the adiabatic change of the Hamiltonian the solution does not exactly follow the spectral subspace $\text{ran } P(t)$, which we will call the adiabatic subspace in the following, but slightly tilted ε -dependent subspaces $\text{ran } P^\varepsilon(t)$, so-called superadiabatic subspaces. With respect to the superadiabatic subspaces one has

$$\|(1 - P^\varepsilon(t))\psi(t)\| = \mathcal{O}(\varepsilon^N) \quad \text{for all } t \in \mathbb{R}. \quad (3)$$

In addition it holds that

$$\|P^\varepsilon(t) - P(t)\| = \mathcal{O}(\varepsilon)$$

which together with (3) implies the usual adiabatic estimate (1). Moreover, at times t when all derivatives of $H(t)$ vanish, the superadiabatic projections agree with the adiabatic projection, $P^\varepsilon(t) = P(t)$, and thus (3) implies also (2). In Figure 1 the situation is plotted schematically.

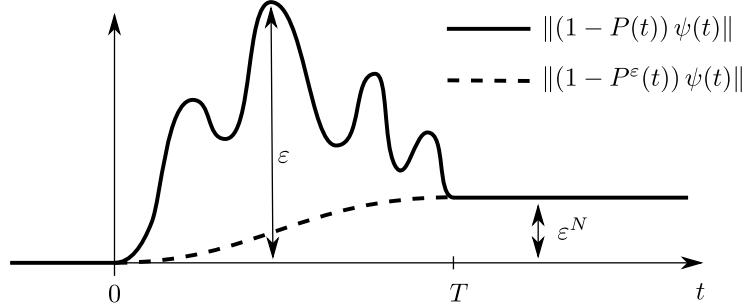


Figure 1: Schematic plot of the transition histories with respect to the adiabatic (full line) and the superadiabatic (dashed line) subspaces. The Hamiltonian varies only during the time interval $[0, T]$ and outside of this interval the adiabatic subspace $P(t)$, i.e. the instantaneous spectral subspace, agrees with the superadiabatic subspace $P^\varepsilon(t)$. While the non-adiabatic transitions for $t \geq T$ are of order ε^N , the norm of the piece of the solution that leaks into the orthogonal complement of $P(t)$ for $0 < t < T$ is typically much larger, namely of order ε . The norm of the piece of the solution that leaks into the orthogonal complement of the superadiabatic subspace $P^\varepsilon(t)$ for $0 < t < T$ remains small, i.e. of order ε^N , for all times.

In this sense we say that the error of the adiabatic approximation is of order ε , while the non-adiabatic transitions are (at most) of order ε^N . For a detailed exposition of adiabatic theory with spectral gap we refer to [18] and references therein. Here let us mention only that the notion “superadiabatic” was termed by M. Berry in [4]. There he shows how for Hamiltonians depending analytically on time the transitions between superadiabatic subspaces are even exponentially small in ε (as it is known also from the famous Landau-Zener model) and moreover, that the transitions as a function of time follow the universal shape of an error function, i.e. an integrated Gaussian. A rigorous account of [4] is given in [6, 7, 8].

In the present paper we show within a physically relevant but relatively simple example that in the gapless case the situation is to some extent similar. Before presenting our results in detail let us introduce the model and explain some of its important features.

We consider a scalar massless field in three spatial dimensions. The momentum space for a single boson is $L^2(\mathbb{R}_k^3)$ and the state space of the field is the symmetric Fock space

$$\mathcal{F} := \bigoplus_{n=0}^{\infty} L^2(\mathbb{R}^3)^{\otimes_s n}.$$

So $\psi \in \mathcal{F}$ is a sequence $\psi = (\psi_0, \psi_1, \psi_2, \dots)$ with $\psi_n(k_1, \dots, k_n)$ a square integrable symmetric function of n variables in \mathbb{R}^3 . For $\psi, \phi \in \mathcal{F}$ the inner product is

$$\langle \psi, \phi \rangle_{\mathcal{F}} = \sum_{n=0}^{\infty} \langle \psi_n, \phi_n \rangle_{L^2(\mathbb{R}^{3n})}.$$

The Hamiltonian of the free field is $H_f := d\Gamma(|k|)$ and acts as

$$(H_f \psi)_n(k_1, \dots, k_n) = \sum_{i=1}^n |k_i| \psi_n(k_1, \dots, k_n).$$

As a multiplication operator H_f is self-adjoint on its dense maximal domain $D(H_f)$. Moreover, H_f has a unique ground state given by the Fock vacuum $\Omega_0 := (1, 0, 0, \dots)$. Now we add moving charges as sources to the field. For notational simplicity we assume that all sources have the same normalized form factor $\varphi : \mathbb{R}^3 \rightarrow [0, \infty)$ with

$$\int_{\mathbb{R}^3} dx \varphi(x) = 1.$$

Later on we will make additional assumptions on the “charge distribution” φ . The sources are located at positions $x_j \in \mathbb{R}^3$ with total charge e_j and form factor φ . The linear coupling to the field is given by the operator

$$H_1(x) := \sum_{j=1}^N H_{1,j}(x_j) := \sum_{j=1}^N e_j \Phi\left(\frac{\hat{\varphi}(k)}{\sqrt{|k|}} e^{ik \cdot x_j}\right),$$

where we abbreviate $x = (x_1, \dots, x_N)$. Here $\hat{\varphi}$ denotes the Fourier transform of φ and Φ is the field operator

$$\Phi(f) := \frac{1}{\sqrt{2}}(a^\dagger(f) + a(f)),$$

where a and a^\dagger are the standard bosonic annihilation and creation operators on \mathcal{F} , see Section 3.1 for more details. Under appropriate conditions on φ the total Hamiltonian $H(x) = H_f + H_1(x)$ is self-adjoint on $D(H) := D(H_f)$ and bounded from below. If the total charge of the sources $e = \sum_{j=1}^N e_j$ is zero, $H(x)$ has a unique ground state $\Omega(x) \neq \Omega_0$. The ground state $\Omega(x)$ contains so-called “virtual bosons” that provide a “dressing” of the sources. In contrast to “free bosons”, the virtual bosons do not propagate. These heuristic notions will be made more precise later on. If the total charge is different from zero there is still a good notion of approximate ground state, at least for our purposes.

We will be interested in the dynamics of the scalar field when the sources change their locations with time. Let $x_j : \mathbb{R} \rightarrow \mathbb{R}^3$, $t \mapsto x_j(t)$ be smooth, then

$$H(t) := H(x(t)) = H_f + \sum_{j=1}^N H_{1,j}(x_j(t))$$

describes the evolution of the scalar field in the presence of sources moving along the prescribed trajectories $x_j(t)$. We consider the solutions of the Schrödinger equation

$$i\varepsilon \frac{d}{dt} \psi(t) = H(t) \psi(t)$$

for the field in the adiabatic limit $\varepsilon \ll 1$, which corresponds to slowly moving sources. Assume for a moment that the field is restricted to a finite box. Then the Hamiltonian $H(t)$ has purely discrete spectrum and the ground state is separated from the first excited state by a gap. The adiabatic

theorem implies that if the field starts in the ground state $\Omega(x(0))$ of $H(0)$, it remains up to a phase ε -close to the ground state $\Omega(x(t))$ of $H(t)$ at later times. The solution is thus ε -close to a state where a static cloud of virtual bosons follows the slowly moving sources and no radiation is emitted. The error of the adiabatic approximation is of order ε . If we pass to the superadiabatic approximation, we realize that the solution is even ε^N -close to a state $\Omega^\varepsilon(t)$ where an ε -dependent cloud of virtual bosons follows the slowly moving sources and no radiation is emitted. The state $\Omega^\varepsilon(t)$ is still ε -close to the static ground state $\Omega(x(t))$, however, the dressing by virtual bosons depends now also on the velocities $\dot{x}_j(t)$ and higher derivatives of $x_j(t)$. If at some time the sources come to rest then from that time on $\Omega^\varepsilon(t) = \Omega(x(t))$, hence the field is ε^N -close to the static ground state again. In particular, the probability for emitting a free boson is at most of order ε^{2N} .

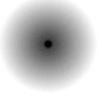
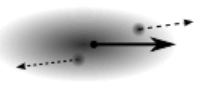
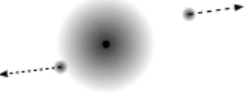

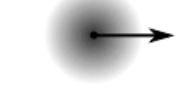

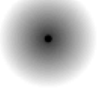
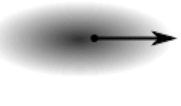

	$t \leq 0$	$0 < t < T$	$t \geq T$
$\psi(t)$			
$\psi_{\text{ad}}(t)$			
$\ \Delta_{\text{ad}}(t)\ =$	0	$\mathcal{O}(\varepsilon)$	$\mathcal{O}(\varepsilon^N)$
$\psi_{\text{su}}(t)$			
$\ \Delta_{\text{su}}(t)\ =$	0	$\mathcal{O}(\varepsilon^N)$	$\mathcal{O}(\varepsilon^N)$

Figure 2: The situation with spectral gap: For $t \leq 0$ the sources are at rest and the adiabatic approximation agrees with the true solution. While the sources are moving, the dressing is deformed, which leads to an error of order ε in the adiabatic approximation. The superadiabatic approximation takes into account the deformation and the error is only of order ε^N . When the sources are again at rest for $t \geq T$, the dressing is again the static one and the error of the adiabatic and the superadiabatic approximation is of order ε^N . This error is due to non-adiabatic transitions that correspond to the emission of free bosons.

Summarizing the above, we have that as long as there is a gap and as long as the prescribed trajectories $x_j(t)$ are $C^{N+1}(\mathbb{R}, \mathbb{R}^3)$, non-adiabatic transitions corresponding to the emission of free bosons are at most of order ε^N . Nonetheless, the error of the adiabatic approximation is of order ε for all times where $\dot{x}_j(t) \neq 0$ because of the deformation of the cloud of virtual bosons, see also Figure 2. This exemplifies in a simple physical example why it is important to distinguish carefully between the error of the adiabatic approximation and the non-adiabatic transitions. It also is an example where superadiabatic subspaces have a clear physical meaning as velocity-dependent dressing of the sources. This is of course all well known and the content of this paper is to show

that this picture survives also in the case without spectral gap, i.e. in the thermodynamic limit where the box is replaced by \mathbb{R}^3 .

In the remainder of the introduction we provide informal statements of our main results, where for simplicity we restrict to the case of a neutral system. Then there is a unitary transformation $V(x) \in \mathcal{L}(\mathcal{F})$ such that $V(x)H(x)V(x)^* = H_f + E(x)$ and thus $H(x)$ has a unique ground state $\Omega(x) = V(x)^*\Omega_0$ with eigenvalue $E(x)$. The general and rigorous statements are explained in Section 2. Our first result is an adiabatic theorem without spectral gap, cf. Theorem 1.

Adiabatic Theorem: *The solution of*

$$i\varepsilon \frac{d}{dt}\psi(t) = H(x(t))\psi(t) \quad (4)$$

with $\psi(0) = \Omega(x(0))$ satisfies for any $t \in \mathbb{R}$

$$\|\psi(t) - e^{-\frac{i}{\varepsilon} \int_0^t ds E(x(s))} \Omega(x(t))\| = \mathcal{O}\left(\varepsilon \sqrt{\ln(1/\varepsilon)}\right). \quad (5)$$

Thus, although the spectrum of $H(x)$ is the whole half line $[E(x), \infty)$ and the eigenvalue $E(x)$ is not separated by a gap from the continuous spectrum, the usual adiabatic approximation for the evolution of eigenstates holds with almost the same error estimate as in the gapped case. We have even more: if one adds free bosons then their dynamics is adiabatic too in the following sense. To each configuration x of the sources the annihilation and creation operators of free bosons are $b_x := V(x)^* a V(x)$ and $b_x^\dagger := V(x)^* a^\dagger V(x)$ and thus

$$e^{\frac{i}{\varepsilon} H(x)t} b_x^\#(f) e^{-\frac{i}{\varepsilon} H(x)t} = b_x^\#(f(t)),$$

where $f(k, t) = e^{-\frac{i}{\varepsilon} |k|t} f(k)$ is just the free time evolution of a boson and $b_x^\#$ stands either for b_x or b_x^\dagger . Now we define the adiabatic approximation as

$$\psi_{\text{ad}}(t) := e^{-\frac{i}{\varepsilon} \int_0^t ds E(x(s))} \prod_{l=1}^m b_{x(t)}^\dagger(f_l(t)) \Omega(x(t)).$$

We will show that it approximates the solution of (4) with initial datum

$$\psi(0) = \prod_{l=1}^m b_{x(0)}^\dagger(f_l) \Omega(x(0)), \quad f_l \in L^2(\mathbb{R}^3),$$

in the same sense as in (5),

$$\|\psi(t) - \psi_{\text{ad}}(t)\| = \mathcal{O}\left(\varepsilon \sqrt{\ln(1/\varepsilon)}\right). \quad (6)$$

If we denote by $Q_m(x) := V(x)^* Q_m V(x)$ the projection on the sector of Fock space containing exactly $m \in \mathbb{N}_0$ free bosons, where Q_m is the projection on the m -particle sector of Fock space, the above result implies that for $\psi(0) \in Q_m(x(0))\mathcal{F}$

$$\|(1 - Q_m(x(t)))\psi(t)\|^2 = \mathcal{O}(\varepsilon^2 \ln(1/\varepsilon)),$$

i.e. that the probability for emitting a free boson is at most of order $\varepsilon^2 \ln(1/\varepsilon)$. Note that this means that not only the spectral subspace $Q_0(x(t))\mathcal{F}$ is adiabatically invariant, but also the subspaces $Q_m(x(t))\mathcal{F}$, which are not spectral subspaces of the instantaneous Hamiltonian $H(x(t))$.

In order to understand what part of $(1 - Q_m(x(t)))\psi(t)$ really corresponds to emission of free bosons and what part is merely a velocity-dependent deformation of the dressing, we need to introduce the superadiabatic picture. Now not only the dressing of the sources depends on their velocities, but also the annihilation and creation operators of free bosons and the associated m -particle sectors of Fock space. In Lemma 9 we construct a velocity-dependent dressing operator

$V_\sigma^\varepsilon(t)$ such that the corresponding objects $\Omega^\varepsilon(t) := V^{\varepsilon*}(t)\Omega_0$, $b_t^{\varepsilon\#} := V^{\varepsilon*}(t)a^\#V_\sigma^\varepsilon(t)$ and $Q_m^\varepsilon(t) := V^{\varepsilon*}(t)Q_mV_\sigma^\varepsilon(t)$ allow for an improved adiabatic approximation. Let

$$\psi_{\text{su}}(t) := e^{-\frac{i}{\varepsilon} \int_0^t ds E^\varepsilon(x(s))} \prod_{l=1}^m b_l^{\varepsilon\#}(f_l(t)) \Omega^\varepsilon(t)$$

be the superadiabatic approximation to the solution $\psi(t)$ of (4) with initial datum

$$\psi(0) = \prod_{l=1}^m b_0^{\varepsilon\#}(f_l) \Omega^\varepsilon(0), \quad f_l \in L^2(\mathbb{R}^3),$$

then (6) can be improved in the following sense: the part of the solution $\psi(t)$ that stays in the m -particle sector is given by $\psi_{\text{su}}(t)$ with a much smaller error,

$$\|Q_m^\varepsilon(t)\psi(t) - \psi_{\text{su}}(t)\| = \mathcal{O}(\varepsilon^2 \ln(1/\varepsilon)).$$

However, more importantly, the superadiabatic representation can be used to compute the non-adiabatic transitions corresponding to the emission of free bosons by first order perturbation theory. In Theorem 2 we show that the “non-adiabatic” wave function

$$\psi_{\text{na}}(t) := -i\varepsilon \int_0^t ds e^{-\frac{i}{\varepsilon} \int_s^t d\tilde{s} E^\varepsilon(x(\tilde{s}))} \Phi_s^\varepsilon \left(e^{-\frac{i}{\varepsilon} |k|t} g^\varepsilon(s) \right) \psi_{\text{su}}(s)$$

satisfies

$$\|(1 - Q_m^\varepsilon(t))\psi(t) - \psi_{\text{na}}(t)\| = \mathcal{O}(\varepsilon^2 \ln(1/\varepsilon)).$$

Here $\Phi_s^\varepsilon(f) := \frac{1}{\sqrt{2}} (b_s^{\varepsilon\#}(f) + b_s^\varepsilon(f))$ and $g^\varepsilon(s) = g^\varepsilon(x(s), \ddot{x}(s))$ is a coupling function depending on the acceleration of the sources. In summary

$$\|\psi(t) - (\psi_{\text{su}}(t) + \psi_{\text{na}}(t))\| = \mathcal{O}(\varepsilon^2 \ln(1/\varepsilon))$$

and thus $\psi_{\text{su}}(t) + \psi_{\text{na}}(t)$ yields a good approximation of the true solution $\psi(t)$ with a clear separation of a “superadiabatic” piece $\psi_{\text{su}}(t)$ that improves the adiabatic approximation and a “non-adiabatic” piece $\psi_{\text{na}}(t)$ that contains real non-adiabatic transitions. Note that a posteriori one can compute the leading order of $\psi_{\text{na}}(t)$ replacing all superadiabatic objects by their leading order adiabatic counterparts,

$$\psi_{\text{na}}(t) = -i\varepsilon \int_0^t ds e^{-\frac{i}{\varepsilon} \int_s^t d\tilde{s} E(x(\tilde{s}))} \Phi_s \left(e^{-\frac{i}{\varepsilon} |k|t} g^\varepsilon(s) \right) \psi_{\text{ad}}(s) + \mathcal{O}(\varepsilon^2 \ln(1/\varepsilon)).$$

Finally one can ask for the probability of emitting a free boson, i.e. for computing $\|\psi_{\text{na}}(t)\|^2$. However, due to the infrared problem it turns out that this has no nice asymptotics for $\varepsilon \rightarrow 0$ and we can only show $\|\psi_{\text{na}}(t)\|^2 = \mathcal{O}(|t|^2 \varepsilon^2 \ln(1/\varepsilon))$. But a physically more interesting question is the energy radiated through the emission of free bosons. In Theorem 3 we show that when starting in the dressed vacuum, i.e. $\psi(0) = \Omega^\varepsilon(0)$, then

$$E_{\text{rad}}(t) := \langle \psi_{\text{na}}(t), H_{\text{f}}(t) \psi_{\text{na}}(t) \rangle_{\mathcal{F}} = \frac{\varepsilon^3}{2 \cdot 6\pi} \int_0^t ds |\ddot{d}(s)|^2 + o(\varepsilon^3),$$

where

$$\ddot{d}(t) := \sum_{j=1}^N e_j \ddot{x}_j(t)$$

is the second derivative of the dipole moment of the sources and $H_{\text{f}}(t)$ is the field energy of the free photons. Taking a derivative gives the Larmor formula for the radiated power of slowly moving charges,

$$P_{\text{rad}}(t) := \frac{d}{dt} E_{\text{rad}}(t) = \frac{\varepsilon^3}{2 \cdot 6\pi} |\ddot{d}(t)|^2 + o(\varepsilon^3).$$

To come back to our main message once more, note that computing naively the energy in the piece of the solution that constitutes the error of the adiabatic approximation $\tilde{\psi}(t) := \psi(t) - \psi_{\text{ad}}(t)$ would have led to

$$\langle \tilde{\psi}(t), H_{\text{f}}(t) \tilde{\psi}(t) \rangle_{\mathcal{F}} = \frac{\varepsilon^2}{4} \sum_{i,j=1}^N \int_{\mathbb{R}^3} dk \frac{e_i e_j |\hat{\varphi}(k)|^2}{|k|^2} e^{ik \cdot (x_j(t) - x_i(t))} \kappa \cdot \dot{x}_j(t) \kappa \cdot \dot{x}_i(t) + \mathcal{O}(\varepsilon^3),$$

where $\kappa := k/|k|$. As our analysis showed, this velocity-dependent term is the energy in the deformation of the dressing and being of order ε^2 , it is much bigger than the energy emitted by radiation of free bosons, which is of order ε^3 . In order to obtain the correct picture, the use of superadiabatic approximations seems inevitable.

Finally let us mention that a closely related problem was considered in [16], see also [15]. There semiclassical limit of non-relativistic Schrödinger particles coupled to a massless scalar field was considered. In a sense the system considered in the present paper can be considered a toy model version of the one in [16]. However, as a consequence of the simplicity of the model the results obtained here are much stronger and at the same time the proofs are more transparent. For

Acknowledgements. We thank Rainer Nagel for pointing out to us reference [11].

2 Main results

We always assume the following properties for the parameters of the model:

Assumptions. *The charge distribution $\varphi \in S(\mathbb{R}^3)$ is a spherically symmetric Schwartz-function with $\hat{\varphi}(0) = (2\pi)^{-3/2}$ and $x_j \in C^4(\mathbb{R}, \mathbb{R}^3)$ for $j = 1, \dots, N$.*

As explained in the introduction, the central object that we construct is the transformation to the superadiabatic representation, a unitary dressing transformation

$$V^\varepsilon(t) : \mathcal{F} \rightarrow \mathcal{F}.$$

In the superadiabatic representation the Schrödinger equation reads

$$V^\varepsilon(t) \left(i\varepsilon \frac{d}{dt} - H(x(t)) \right) V^\varepsilon(t)^* V^\varepsilon(t) \psi(t) =: \left(i\varepsilon \frac{d}{dt} - H_{\text{dress}}^\varepsilon(t) \right) \phi(t).$$

In this representation the Fock vacuum Ω_0 corresponds to the dressed ground state and the m -particle sectors of Fock space correspond to states with m free bosons. While in the introduction we formulated the statements in the original representation, it is more convenient to formulate and prove everything in the new representation after performing the dressing transformation. The translation back to the original representation is straightforward using the definitions of $b_t^{\varepsilon\#}$ of the introduction. Note, however, that if the total charge is different from zero, the transformation $V^\varepsilon(t)$ has no asymptotic expansion in powers of ε , not even the limit $\lim_{\varepsilon \rightarrow 0} V^\varepsilon(t)$ exists.

Let $Q_{\leq M}$ denote the projection on the sectors of Fock space with at most M photons.

Theorem 1 (Adiabatic approximation). *The dressed Hamiltonian $H_{\text{dress}}^\varepsilon$ generates a unitary propagator $U_{\text{dress}}(t) := U_{\text{dress}}(t, t_0 = 0)$. For any $T \in \mathbb{R}$ there are constants $C < \infty$ and $\varepsilon_0 > 0$ such that for all $M \in \mathbb{N}$, $0 < \varepsilon \leq \varepsilon_0$ and $|t| \leq T$*

$$\left\| \left(U_{\text{dress}}(t) - e^{-\frac{i}{\varepsilon} (H_{\text{f}} t + \int_0^t E(s) ds)} \right) Q_{\leq M} \right\| \leq C \varepsilon \ln(\varepsilon^{-1}) \sqrt{M+1}$$

in the norm of $\mathcal{L}(\mathcal{F})$ and in the norm of $\mathcal{L}(D(H_{\text{f}}))$. Here

$$E(t) := -\frac{1}{2} \int_{\mathbb{R}^3} dk \frac{|v_0(x(t), k)|^2}{|k|} \quad (7)$$

with

$$v_0(x(t), k) := \sum_{j=1}^N \frac{e_j \hat{\varphi}(k)}{|k|^{1/2}} e^{ik \cdot x_j(t)}. \quad (8)$$

Thus at leading order the time evolution is just the free evolution of the bosons and a time-dependent dynamical phase. In particular, the m -boson sectors $Q_m \mathcal{F}$ are adiabatic invariants of the dynamics. An initial state in $Q_m \mathcal{F}$ of the form

$$\phi(0) = \prod_{l=1}^m a^\dagger(f_l) \Omega_0, \quad f_l \in L^2(\mathbb{R}^3),$$

evolves into

$$\phi(t) := U_{\text{dress}}(t) \phi(0) = e^{-\frac{i}{\varepsilon} \int_0^t ds E(s)} \prod_{l=1}^m a^\dagger(f_l(t)) \Omega_0.$$

Applying $V^\varepsilon(t)$ and using

$$\|V^\varepsilon(t) - V(x(t))\| = \mathcal{O}(\varepsilon)$$

in the case of a neutral system yields (6).

Theorem 2 (Superadiabatic approximation). *For any $T \in \mathbb{R}$ there are constants $C < \infty$ and $\varepsilon_0 > 0$ such that for all $M \in \mathbb{N}$, $0 < \varepsilon \leq \varepsilon_0$ and $|t| \leq T$*

$$\left\| \left(U_{\text{dress}}(t) - e^{-\frac{i}{\varepsilon} (H_{\text{f}} t + \int_0^t E^\varepsilon(s) ds)} \left(1 + i\varepsilon \int_0^t ds e^{iH_{\text{f}} \frac{s}{\varepsilon}} h_{\text{rad}}(s) e^{-iH_{\text{f}} \frac{s}{\varepsilon}} \right) \right) Q_{\leq M} \right\| \leq C \varepsilon^2 \ln(\varepsilon^{-1}) \sqrt{M+1}$$

in $\mathcal{L}(\mathcal{F})$ and $\mathcal{L}(D(H_{\text{f}}))$ with

$$E^\varepsilon(t) = E(t) + \frac{\varepsilon^2}{4} \sum_{j,i=1}^N \int_{\mathbb{R}^3} dk \frac{e_j e_i |\hat{\varphi}(k)|^2}{|k|^2} e^{-ik \cdot (x_i(t) - x_j(t))} (\kappa \cdot \dot{x}_j(t)) (\kappa \cdot \dot{x}_i(t))$$

and

$$h_{\text{rad}}(t) := \Phi \left(\mathbf{1}_{[\varepsilon^8, \infty)}(|k|) \sum_{j=1}^N \frac{e_j \hat{\varphi}(k)}{|k|^{\frac{3}{2}}} e^{-ik \cdot x_j(t)} \langle \kappa, \ddot{x}_j(t) \rangle \right).$$

Note the infrared regularization by the characteristic function $\mathbf{1}_{[\varepsilon^8, \infty)}(|k|)$ cutting off boson momenta smaller than ε^8 in this definition, which can be omitted for neutral systems.

With the help of this theorem we can define a decomposition of the wave function into a superadiabatic part ϕ_{su} and the non-adiabatic part ϕ_{na} with different photon number. For a wavefunction $\phi_0 \in Q_m \mathcal{F}$ we define

$$\phi_{\text{su}}(t) := Q_m \phi(t) = e^{-\frac{i}{\varepsilon} (H_{\text{f}} t + \int_0^t E^\varepsilon(s) ds)} \phi_0 + \mathcal{O}(\varepsilon^2 \ln(\varepsilon^{-1}))$$

and

$$\phi_{\text{na}}(t) := Q_m^\perp \phi(t) = i\varepsilon e^{-\frac{i}{\varepsilon} (H_{\text{f}} t + \int_0^t E^\varepsilon(s) ds)} \int_0^t ds e^{iH_{\text{f}} \frac{s}{\varepsilon}} h_{\text{rad}}(s) e^{-iH_{\text{f}} \frac{s}{\varepsilon}} \phi_0 + \mathcal{O}(\varepsilon^2 \ln(\varepsilon^{-1})).$$

Our final result concerns the amount of energy lost by the system due to radiation. For the stationary problem with Hamiltonian $H(x)$ independent of time, the natural definition for the energy carried by free photons in the state $\psi \in D(H_{\text{f}})$ is

$$E_{\text{rad,stat}}(\psi) := \langle \psi, H(x) \psi \rangle - E(x),$$

where $E(x)$ is the ground state energy. For time-dependent Hamiltonians the definition of energy is somewhat subtle, more so, because we look at the energy of a subsystem, the free photons. However,

in the superadiabatic representation there is again a natural definition for the energy given by the free photons, namely

$$E_{\text{rad}}(\psi(t)) := \langle V^\varepsilon(t) \psi, H_f V^\varepsilon(t) \psi \rangle. \quad (9)$$

We will show that at any time t when $\dot{x}(t) = \ddot{x}(t) = 0$, it holds that

$$E_{\text{rad}}(\psi(t)) = E_{\text{rad,stat}}(\psi(t)) + \mathcal{O}(\varepsilon^4). \quad (10)$$

Thus in a situation where the sources move only during a finite time interval, before and after the change the definition of energy agrees with the static one. Note that the error of order ε^4 comes from the infrared regularization in the transformation $V^\varepsilon(t)$ and is identically zero for neutral systems. The equality (10) is also determining the exponent 8 in the infrared regularization. A smaller exponent there would lead to a larger error in (10).

If we assume that the initial state is the dressed vacuum, then the energy emitted as a result of the motion of the sources is just the energy of the free photons and thus $E_{\text{rad}}(t) = \langle \phi_{\text{na}}(t), H_f \phi_{\text{na}}(t) \rangle$, which is computed explicitly in the following theorem.

Theorem 3 (Radiation). *Let $\phi(0) = \Omega$, then uniformly on bounded intervals in time it holds that*

$$E_{\text{rad}}(t) = \frac{\varepsilon^2}{2} \sum_{i,j=1}^N e_j e_i \int dk \frac{|\hat{\varphi}(k)|^2}{|k|^2} \int_0^t \int_0^t ds ds' e^{i|k| \frac{s-s'}{\varepsilon}} e^{-ik \cdot (x_j(s) - x_i(s'))} \quad (11)$$

$$\begin{aligned} & \times \kappa \cdot \ddot{x}_j(s) \kappa \cdot \ddot{x}_i(s') + \mathcal{O}\left(\varepsilon^4 (\ln(\varepsilon^{-1}))^2\right) \\ & = \frac{\varepsilon^3}{2 \cdot 6\pi} \int_0^t ds |\ddot{d}(s)|^2 + o(\varepsilon^3), \end{aligned} \quad (12)$$

where

$$\ddot{d}(t) := \sum_{j=1}^N e_j \ddot{x}_j(t)$$

is the second derivative of the dipole moment.

3 Mathematical details

3.1 Operators on Fock space

In this section we recall the definitions of the operators on Fock space and some of their important properties. Proofs of all claims can be found in [14], Section X.7.

We call \mathcal{F}_{fin} the subspace of the Fock space for which $\psi_m = 0$ for all but finitely many m . The second quantization of a self-adjoint multiplication operator ω with domain $D(\omega) \in L^2(\mathbb{R}^3)$ is defined for $\psi \in \mathcal{F}_{\text{fin}}$ with components $\psi_m \in \otimes_{k=1}^m D(\omega)$ as

$$(\text{d}\Gamma(\omega)\psi)_m(k_1, \dots, k_m) = \sum_{j=1}^m \omega(k_j) \psi_m(k_1, \dots, k_m)$$

and is essentially self-adjoint. In particular, the free field Hamiltonian $H_f := \text{d}\Gamma(|k|)$ is self-adjoint on its maximal domain. The annihilation operator and the creation operator on $\psi \in \mathcal{F}_{\text{fin}}$ are defined by

$$\begin{aligned} (a(f)\psi)_m(k_1, \dots, k_m) &:= \sqrt{m+1} \int_{\mathbb{R}^3} dk \bar{f}(k) \psi_{m+1}(k, k_1, \dots, k_m), \\ (a^\dagger(f)\psi)_m(k_1, \dots, k_m) &:= \frac{1}{\sqrt{m}} \sum_{j=1}^m f(k_j) \psi_{m-1}(k_1, \dots, \tilde{k}_j, \dots, k_m). \end{aligned}$$

Here \tilde{k}_j means that k_j is omitted. They fulfill the canonical commutation relations

$$[a(f), a^\dagger(g)] = \langle f, g \rangle_{L^2(\mathbb{R}^3)}, \quad [a(f), a(g)] = 0, \quad [a^\dagger(f), a^\dagger(g)] = 0.$$

The creation and annihilation operator define the Segal field operator

$$\Phi(f) := \frac{1}{\sqrt{2}}(a(f) + a^\dagger(f)).$$

It is essentially self-adjoint on \mathcal{F}_{fin} . The canonical commutation relations imply

$$[\Phi(f), \Phi(g)] = i \operatorname{Im} \langle f, g \rangle_{L^2(\mathbb{R}^3)} \quad \text{and} \quad [d\Gamma(\omega), i\Phi(f)] = \Phi(i\omega f). \quad (13)$$

3.2 The Hamiltonian

As described in the introduction we consider the Hamiltonian

$$H(t) := H(x(t)) = H_f + \Phi(v_0(x(t), k)) =: H_f + \Phi_0(t)$$

with $v_0(x(t), k)$ defined in (8). As to be stated more precisely in Lemma 6 and after, this Hamiltonian has a ground state only for a neutral system, i.e. if $\sum_{j=1}^N e_j = 0$, otherwise the bottom of the spectrum is not an eigenvalue. To deal with this fact we introduce an infrared cutoff $0 \leq \sigma \leq 1$ in the interaction and put $\hat{\varphi}_\sigma(k) = \mathbf{1}_{[\sigma, \infty)}(|k|)\hat{\varphi}(k)$ and $v_\sigma(x, k) = \mathbf{1}_{[\sigma, \infty)}(|k|)v_0(x, k)$. The resulting truncated Hamiltonian $H^\sigma(t) = H_f + \Phi_\sigma(t)$ is a good approximation to the original Hamiltonian as will be proven in Proposition 11. We emphasize that $H^\sigma(t)$ is used only as a tool in the proofs and the final results of Section 2 hold for the Hamiltonian $H(t) = H^{\sigma=0}(t)$ without infrared cutoff.

Lemma 4. *The operators $H^\sigma(t)$ are self-adjoint on $D(H_f)$ for all t and $\sigma \in [0, 1]$. The graph norms of $H^\sigma(t)$ are all equivalent to the one defined by H_f uniformly in σ .*

Proof. There exists a standard estimate for Φ (cf. e.g. [5] Proposition 1.3.8). Let $\Psi \in D(H_f)$ and $b > 0$, then

$$\|\Phi(f)\psi\|_{\mathcal{F}}^2 \leq b \|H_f \psi\|_{\mathcal{F}}^2 + \left(\frac{1}{b} \|f/\sqrt{|k|}\|_{L^2(\mathbb{R}^3)}^4 + 2 \|f\|_{L^2(\mathbb{R}^3)}^2 \right) \|\psi\|_{\mathcal{F}}^2.$$

Thus $\Phi(f)$ is infinitesimally bounded with respect to H_f if

$$\|f\|_{L^2(\mathbb{R}^3)} + \|f/\sqrt{|k|}\|_{L^2(\mathbb{R}^3)} < \infty. \quad (14)$$

Equation (14) holds uniformly for $f = v_\sigma(x(t), k)$ and $\sigma \in [0, 1]$. Hence $H^\sigma(t)$ is self-adjoint on $D(H_f)$ by the Kato-Rellich Theorem (cf. [14], Theorem X.12) for all $\sigma \in [0, 1]$ and the graph norms are all equivalent. \square

Proposition 5. *Let $t \mapsto x(t) \in C^n(\mathbb{R}, \mathbb{R}^{3N})$, $n \in \mathbb{N}$, and equip $D(H_f)$ with the graph norm. Let $\sigma \in [0, 1]$, then*

1. $H^\sigma \in C_b^n(\mathbb{R}, \mathcal{L}(D(H_f), \mathcal{F}))$
2. H^σ generates a strongly continuous unitary evolution family denoted by $U_{H^\sigma}(t, t_0)$. Moreover, $U_{H^\sigma}(t, t_0)$ is a bounded operator on $D(H_f)$ with

$$\sup_{s \in [t_0, t]} \|U_{H^\sigma}(s, t_0)\|_{\mathcal{L}(D(H_f))} < \infty \quad (15)$$

for any $t \in \mathbb{R}$ and uniformly in σ .

Proof. According to (14) we need to show that

$$\left\| \frac{d^n}{dt^n} v_0(x(t), k) \right\| + \left\| \frac{d^n}{dt^n} v_0(x(t), k) / \sqrt{|k|} \right\| < \infty,$$

which obviously holds under our assumptions. That H and H^σ generate unitary evolution families follows from the first statement and the general result about contraction semigroups (e.g. [14], Theorem X.70). The norm bound $U_{H^\sigma}(t, t_0)$ in $\mathcal{L}(D(H_f))$ is less known and was proved in Theorem 5.1.(e) of [11]. \square

Lemma 6. Let $K = H_f + \Phi(z(k))$ with $z(k) \in L^2(\mathbb{R}^3)$ such that

$$\alpha := -\frac{1}{2} \int_{\mathbb{R}^3} dk \frac{|z(k)|^2}{|k|} < \infty.$$

Then the spectrum of K is given by $[\alpha, \infty)$ and the infimum of the spectrum α is an eigenvalue if and only if

$$\int_{\mathbb{R}^3} dk \frac{|z(k)|^2}{|k|^2} < \infty.$$

In this case the unitary operator $V = e^{-i\Phi(\text{iz}(k)/|k|)}$ is well-defined and $K = V^* H_f V + \alpha$.

Proof. The first part is Proposition 3.10 of [10] and the second part is Proposition 3.13 of the same paper. \square

Corollary 7. The spectrum of $H(t)$ is $\sigma(H(t)) = \sigma_{\text{ac}}(H(t)) = [E(t), \infty)$ with $E(t)$ as in (7).

Corollary 8. For $\sigma > 0$, the infrared regularized Hamiltonian $H^\sigma(t)$ can be written as

$$H^\sigma(t) = V_\sigma^*(t) H_f V_\sigma(t) + E_\sigma(t),$$

where

$$V_\sigma(t) := e^{-i\Phi(\text{iv}_\sigma(x(t), k)/|k|)}.$$

Its only eigenvalue with eigenvector $\Omega_\sigma(t) := V_\sigma^*(t) \Omega_0$ is

$$E_\sigma(t) = E(t) + \mathcal{O}(\sigma). \quad (16)$$

Proof. The last statement is an immediate consequence of Lemma 6 and the fact that

$$\int_{\mathbb{R}^3} dk \frac{|v_\sigma(x(t), k)|^2}{|k|} = \int_{\mathbb{R}^3} dk \frac{|v_0(x(t), k)|^2}{|k|} + \mathcal{O}(\sigma). \quad \square$$

3.3 Superadiabatic perturbation and the dressing operator

In this section we introduce the perturbed projections and the transformation which “diagonalizes” the infrared regularized Hamiltonian up to $\mathcal{O}(\varepsilon^2)$. As explained in the introduction the idea is to modify the adiabatic projections $P_0^m(t) := V_\sigma^*(t) Q_m V_\sigma(t)$ on the m -free-photons subspaces in such a way, that the modified superadiabatic projections $P_1^{m, \varepsilon}(t)$ are higher order adiabatic invariants, i.e. that

$$[i\varepsilon \frac{d}{dt} - H(t), P_1^{m, \varepsilon}(t)] = \mathcal{O}(\varepsilon^2)$$

holds. This construction is by now standard, see [12], and yields

$$P_1^{m, \varepsilon}(t) = V_\sigma^*(t) Q_m V_\sigma(t) + i\varepsilon V_\sigma^*(t) [Q_m, \Phi_2(t)] V_\sigma(t) + \mathcal{O}(\varepsilon^2),$$

where we introduce the shorthands

$$\begin{aligned} \Phi_1(t) &:= \frac{d}{dt} \Phi\left(\frac{\text{iv}_\sigma(x(t), k)}{|k|}\right) = \Phi\left(-|k| \sum_{j=1}^N \frac{e_j \hat{\varphi}_\sigma(k)}{|k|^{3/2}} e^{ik \cdot x_j(t)} \langle \kappa, \dot{x}_j(t) \rangle\right) =: \Phi(z_1(t)) \\ \Phi_2(t) &:= -i[d\Gamma(|k|^{-1}), \Phi_1] = \Phi\left(i \sum_{j=1}^N \frac{e_j \hat{\varphi}_\sigma(k)}{|k|^{3/2}} e^{ik \cdot x_j(t)} \langle \kappa, \dot{x}_j(t) \rangle\right) =: \Phi(z_2(t)). \end{aligned}$$

Here κ is the unit vector in the direction of k . The canonical commutation relations link Φ_1 and Φ_2 in the following way

$$i[d\Gamma(|k|), \Phi_2] = \Phi_1.$$

Now the corresponding dressing operator $V_\sigma^\varepsilon(t)$ that maps $P_1^{m,\varepsilon}(t)$ to Q_m then needs to have the expansion

$$V_\sigma^\varepsilon(t) = (\mathbf{1} + i\varepsilon\Phi_2(t) + \mathcal{O}(\varepsilon^2))V_\sigma(t) \quad \text{and} \quad V_\sigma^\varepsilon(t)^* = V_\sigma^*(t)(\mathbf{1} - i\varepsilon\Phi_2(t) + \mathcal{O}(\varepsilon^2)).$$

This suggests to define

$$V_\sigma^\varepsilon(t) := e^{i\varepsilon\Phi_2(t)}V_\sigma(t).$$

Lemma 9. *For $\sigma \in (0, 1]$ and $\varepsilon \in [0, 1]$ the operator $V_\sigma^\varepsilon(t)$ is unitary, belongs to $\mathcal{L}(D(H_f))$ and satisfies*

$$\|V_\sigma^\varepsilon(t)\|_{\mathcal{L}(D(H_f))} + \|V_\sigma^\varepsilon(t)^*\|_{\mathcal{L}(D(H_f))} \leq C < \infty$$

uniformly in σ and ε .

The map $\mathbb{R} \rightarrow \mathcal{L}(D(H_f), \mathcal{F})$, $t \mapsto V_\sigma^\varepsilon(t)$ is differentiable and

$$\frac{d}{dt}V_\sigma^\varepsilon = \left(-i\Phi_1 + i\varepsilon\dot{\Phi}_2 + \varepsilon[\Phi_2, \Phi_1] - \frac{\varepsilon^2}{2}[\Phi_2, \dot{\Phi}_2]\right)V_\sigma^\varepsilon. \quad (17)$$

Proof. With

$$[H_f, V_\sigma(t)] = V_\sigma(t)(V_\sigma^*(t)H_fV_\sigma(t) - H_f) = V_\sigma(t)(\Phi_0(t) - E_\sigma(t))$$

and

$$\begin{aligned} [H_f, e^{i\varepsilon\Phi_2}] &= e^{i\varepsilon\Phi_2} (e^{-i\varepsilon\Phi_2} H_f e^{i\varepsilon\Phi_2} - H_f) \\ &= e^{i\varepsilon\Phi_2} \left(\sum_{j=0}^{\infty} \varepsilon^j \frac{[-i\Phi_2, [-i\Phi_2, [-i\Phi_2, \dots H_f]]]}{j!} - H_f \right) \\ &= e^{i\varepsilon\Phi_2} \left(\varepsilon\Phi_1 - i\frac{\varepsilon^2}{2}[\Phi_2, \Phi_1] \right) \end{aligned}$$

the first statement follows, since Φ_0 , Φ_1 and $[\Phi_2, \Phi_1] = i\text{Im}\langle z_2, z_1 \rangle$ are bounded independently of σ as operators from $D(H_f)$ to \mathcal{F} .

Under our hypotheses on $\hat{\varphi}$ the operators $\Phi(iv_\sigma(x(t), k)/|k|)$ and $\dot{\Phi} = \Phi_1$ commute as a consequence of (13) and $\langle v_\sigma(x(t), k)/|k|, v_\sigma(x(t), k)\kappa \cdot \dot{x}_j(t) \rangle = 0$. Hence

$$\frac{d}{dt}V_\sigma(t) = \frac{d}{dt}e^{-i\Phi(iv_\sigma(x(t), k)/|k|)} = -i\dot{\Phi}(t)V_\sigma(t) = -i\Phi_1(t)V_\sigma(t).$$

For the other exponential we find

$$\frac{d}{dt}e^{i\varepsilon\Phi_2} = [\partial_t, e^{i\varepsilon\Phi_2}] = (\partial_t - e^{i\varepsilon\Phi_2}\partial_t e^{-i\varepsilon\Phi_2})e^{i\varepsilon\Phi_2} = \left(i\varepsilon\dot{\Phi}_2 - \frac{\varepsilon^2}{2}[\Phi_2, \dot{\Phi}_2]\right)e^{i\varepsilon\Phi_2}.$$

Finally

$$e^{i\varepsilon\Phi_2}\Phi_1 e^{-i\varepsilon\Phi_2} = \Phi_1 + i\varepsilon[\Phi_2, \Phi_1]$$

implies (17). □

3.4 The dressed Hamiltonian

With the help of the dressing transformation $V_\sigma^\varepsilon(t)$, we define the dressed Hamiltonians

$$\begin{aligned} i\varepsilon\partial_t - H_{\text{dress}}(t) &:= V_\sigma^\varepsilon(t)(i\varepsilon\partial_t - H(t))V_\sigma^\varepsilon(t)^*, \\ i\varepsilon\partial_t - H_{\text{dress}}^\sigma(t) &:= V_\sigma^\varepsilon(t)(i\varepsilon\partial_t - H^\sigma(t))V_\sigma^\varepsilon(t)^*. \end{aligned}$$

As a consequence of Lemma 9 the dressed Hamiltonians are self-adjoint on $D(H_f)$ since $V_\sigma^\varepsilon(t)$ is a bijection on $D(H_f)$. The corresponding evolution families are related through

$$U_{\text{dress}}(t) = V_\sigma^\varepsilon(t)U_H(t)V_\sigma^\varepsilon(0)^* \quad \text{and} \quad U_{\text{dress}}^\sigma(t) = V_\sigma^\varepsilon(t)U_{H^\sigma}(t)V_\sigma^\varepsilon(0)^*.$$

Lemma 10. *The dressed Hamiltonian has the form*

$$H_{\text{dress}}^\sigma(t) = H_f + E_\sigma^\varepsilon(t) - \varepsilon^2 \dot{\Phi}_2(t)$$

with

$$E_\sigma^\varepsilon(t) := E_\sigma(t) - \frac{\varepsilon^2}{2} \text{Im} \langle z_2(t), z_1(t) \rangle + \frac{\varepsilon^3}{2} \text{Im} \langle z_2(t), \dot{z}_2(t) \rangle.$$

Proof. For $H^\sigma(t)$ we find

$$\begin{aligned} V_\sigma^\varepsilon H^\sigma V_\sigma^{\varepsilon*} &= e^{i\varepsilon\Phi_2} V_\sigma H^\sigma V_\sigma^* e^{-i\varepsilon\Phi_2} = e^{i\varepsilon\Phi_2} (H_f + E_\sigma) e^{-i\varepsilon\Phi_2} \\ &= \sum_{j=0}^{\infty} \varepsilon^j \frac{[i\Phi_2, [i\Phi_2, [i\Phi_2, \dots H_f]]]}{j!} + E_\sigma \\ &= H_f + E_\sigma - \varepsilon\Phi_1 - i\frac{\varepsilon^2}{2} [\Phi_2, \Phi_1] = H_f + E_\sigma - \varepsilon\Phi_1 + \frac{\varepsilon^2}{2} \text{Im} \langle z_2, z_1 \rangle, \end{aligned}$$

and with (17)

$$V_\sigma^\varepsilon i\varepsilon \partial_t V_\sigma^{\varepsilon*} = i\varepsilon \partial_t - i\varepsilon \left(\frac{d}{dt} V^\varepsilon \right) V^{\varepsilon*} = i\varepsilon \partial_t - \varepsilon\Phi_1 + \varepsilon^2 \dot{\Phi}_2 - i\varepsilon^2 [\Phi_2, \Phi_1] + \frac{i\varepsilon^3}{2} [\Phi_2, \dot{\Phi}_2].$$

Hence

$$H_{\text{dress}}^\sigma(t) = H_f + E_\sigma(t) - \varepsilon^2 \dot{\Phi}_2(t) - \frac{\varepsilon^2}{2} \text{Im} \langle z_2(t), z_1(t) \rangle + \frac{\varepsilon^3}{2} \text{Im} \langle z_2(t), \dot{z}_2(t) \rangle. \quad \square$$

Proposition 11. *For any $\sigma, \varepsilon \in (0, 1]$ and $M \in \mathbb{N}$ we have*

$$\left\| \left(U_{\text{dress}}(t) - U_{\text{dress}}^\sigma(t) \right) Q_{\leq M} \right\| = \mathcal{O} \left(\sqrt{M+1} \frac{\sigma}{\varepsilon} \right)$$

uniformly on bounded intervals in time in $\mathcal{L}(\mathcal{F})$ and $\mathcal{L}(D(H_f))$.

Proof. By definition we have

$$\left(U_{\text{dress}}(t) - U_{\text{dress}}^\sigma(t) \right) Q_{\leq M} = V_\sigma^\varepsilon(t) (U_H(t) - U_{H^\sigma}(t)) V_\sigma^\varepsilon(0)^* Q_{\leq M}.$$

Let

$$H(s) - H^\sigma(s) = \Phi(\mathbf{1}_{(0,\sigma)}(|k|) v_0(x(s), k)) =: \Phi(f_{\leq \sigma}(s)).$$

Then bosons with momenta smaller than σ evolve freely under the regularized evolution, i.e.

$$U_{H^\sigma}^*(t) \Phi(e^{i|k|\frac{t}{\varepsilon}} f_{\leq \sigma}(s)) U_{H^\sigma}(t) = \Phi(f_{\leq \sigma}(s))$$

holds for all $t \in \mathbb{R}$. To see this note that the equality holds for $t = 0$ and the derivative of the left hand side vanishes since

$$[H^\sigma, \Phi(e^{i|k|\frac{t}{\varepsilon}} f_{\leq \sigma}(s))] = [H_f, \Phi(e^{i|k|\frac{t}{\varepsilon}} f_{\leq \sigma}(s))] = -i\varepsilon \frac{d}{dt} \Phi(e^{i|k|\frac{t}{\varepsilon}} f_{\leq \sigma}(s)).$$

Moreover, $\Phi(f_{\leq \sigma}(s))$ commutes with $V_\sigma^\varepsilon(0)^*$. By the Duhamel formula we thus find

$$\begin{aligned} \|(U_H(t) - U_{H^\sigma}(t)) V_\sigma^\varepsilon(0)^* Q_{\leq M}\| &\leq \frac{1}{\varepsilon} \int_0^t ds \|(H(s) - H^\sigma(s)) U_{H^\sigma}(s) V_\sigma^\varepsilon(0)^* Q_{\leq M}\| \\ &= \frac{1}{\varepsilon} \int_0^t ds \|\Phi(f_{\leq \sigma}(s)) U_{H^\sigma}(s) V_\sigma^\varepsilon(0)^* Q_{\leq M}\| \\ &= \frac{1}{\varepsilon} \int_0^t ds \left\| U_{H^\sigma}(s) \Phi(-e^{i|k|\frac{s}{\varepsilon}} f_{\leq \sigma}(s)) V_\sigma^\varepsilon(0)^* Q_{\leq M} \right\| \\ &= \frac{1}{\varepsilon} \int_0^t ds \left\| \Phi(-e^{i|k|\frac{s}{\varepsilon}} f_{\leq \sigma}(s)) Q_{\leq M} \right\|. \end{aligned}$$

The following lemma, which will be applied several times in the following, together with the fact that

$$\|f_{\leq \sigma}(s)\|_{L^2} + \|k|f_{\leq \sigma}(s)\|_{L^2} = \mathcal{O}(\sigma)$$

uniformly for $s \in [0, t]$ allows us to conclude. \square

Lemma 12. *There is a constant $C < \infty$ such that for any $f \in L^2(\mathbb{R}^3)$ and $M \in \mathbb{N}$*

$$\|\Phi(f) Q_{\leq M}\|_{\mathcal{L}(\mathcal{F})} \leq 2^{\frac{1}{2}} \sqrt{M+1} \|f\|_{L^2(\mathbb{R}^3)}$$

and

$$\|\Phi(f) Q_{\leq M}\|_{\mathcal{L}(D(H_f))} \leq C \sqrt{M+1} \left(\|f\|_{L^2(\mathbb{R}^3)} + \| |k| f \|_{L^2(\mathbb{R}^3)} \right)$$

Proof. According to [14], Theorem X.41, we have for $f \in L^2(\mathbb{R}^3)$ that

$$\|\Phi(f) Q_{\leq M} \psi\|_{\mathcal{F}} \leq 2^{\frac{1}{2}} \sqrt{M+1} \|f\|_{L^2(\mathbb{R}^3)} \|\psi\|_{\mathcal{F}}.$$

The second claim follows from the observation that

$$H_f \Phi(f) Q_{\leq M} = \Phi(f) Q_{\leq M} H_f - i \Phi(|k| f) Q_{\leq M}$$

together with the first estimate. \square

3.5 Effective dynamics

In this section we first show that the statements of Theorems 1 and 2 hold for the infrared regularized evolution $U_{\text{dress}}^\sigma(t)$ with an error depending only logarithmically on σ . Then we use Proposition 11 and an appropriate choice for $\sigma = \sigma(\varepsilon)$ to show the statements also for the full evolution $U_{\text{dress}}(t)$. Clearly the statement of Theorem 1 is a consequence of Theorem 2, but since the proof necessarily proceeds in the same two steps, we separated the statements.

Lemma 13. *For any $0 < \sigma \leq \frac{1}{3}$ and $M \in \mathbb{N}$ it holds that $\dot{\Phi}_2 Q_{\leq M} \in \mathcal{L}(D(H_f)) \cap \mathcal{L}(\mathcal{F})$ with*

$$\|\dot{\Phi}_2 Q_{\leq M}\| = \mathcal{O}(\sqrt{M+1} \ln(\sigma^{-1}))$$

uniformly on bounded time intervals in $\mathcal{L}(\mathcal{F})$ and $\mathcal{L}(D(H_f))$.

Proof. Since

$$\|\dot{z}_2\|_{L^2(\mathbb{R}^3)}^2 = \int_{|k|>\sigma} dk \left| \sum_{j=1}^N \frac{e_j \hat{\varphi}(k)}{|k|^{\frac{1}{2}}} e^{-ik \cdot x_j} \left(\frac{1}{|k|} \langle \kappa, \ddot{x}_j \rangle + \langle \kappa, \dot{x}_j \rangle^2 \right) \right|^2 = \mathcal{O}(\ln(\sigma^{-1})), \quad (18)$$

the claims follow from Lemma 12. \square

Proposition 14. *For $\sigma, \varepsilon \in (0, \frac{1}{3}]$ and $M \in \mathbb{N}$ we have that*

$$\left\| \left(U_{\text{dress}}^\sigma(t) - e^{-\frac{1}{\varepsilon}(H_f t + \int_0^t E_\sigma^\varepsilon(s) ds)} \right) Q_{\leq M} \right\| = \mathcal{O}(\varepsilon \sqrt{M+1} \ln(\sigma^{-1}))$$

uniformly on bounded time intervals in $\mathcal{L}(\mathcal{F})$ and $\mathcal{L}(D(H_f))$.

Proof. Note that (15) together with Lemma 9 implies $U_{\text{dress}}^\sigma(s) \in \mathcal{L}(D(H_f))$ uniformly for $s \in [0, t]$. With the shorthand $U_0(t) := e^{-\frac{1}{\varepsilon}(H_f t + \int_0^t E_\sigma^\varepsilon(s) ds)}$ and the previous lemma, we thus find

$$\begin{aligned} U_{\text{dress}}^\sigma(t) Q_{\leq M} &= U_0(t) Q_{\leq M} + \frac{i}{\varepsilon} U_{\text{dress}}^\sigma(t) \int_0^t ds U_{\text{dress}}^{\sigma *} (s) \varepsilon^2 \dot{\Phi}_2(t) U_0(t) Q_{\leq M} \\ &= U_0(t) Q_{\leq M} + i \varepsilon U_{\text{dress}}^\sigma(t) \int_0^t ds U_{\text{dress}}^{\sigma *} (s) \dot{\Phi}_2(t) Q_{\leq M} U_0(t) \\ &= U_0(t) Q_{\leq M} + \mathcal{O}(\varepsilon \ln(\sigma^{-1})). \end{aligned} \quad \square$$

For the next proposition we abbreviate

$$U_{\text{eff}}^\sigma(t) := e^{-\frac{i}{\varepsilon}(H_f t + \int_0^t E_\sigma^\varepsilon(s) ds)} \left(1 + i\varepsilon \int_0^t ds e^{iH_f \frac{s}{\varepsilon}} h_{\text{rad}}^\sigma(s) e^{-iH_f \frac{s}{\varepsilon}} \right)$$

with

$$h_{\text{rad}}^\sigma(t) := \Phi \left(\sum_{j=1}^N \frac{e_j \hat{\varphi}_\sigma(k)}{|k|^{\frac{3}{2}}} e^{-ik \cdot x_j(t)} \langle \kappa, \ddot{x}_j(t) \rangle \right).$$

Proposition 15. *For $\sigma, \varepsilon \in (0, \frac{1}{3}]$ and $M \in \mathbb{N}$ we have that*

$$\left\| \left(U_{\text{dress}}^\sigma(t) - U_{\text{eff}}^\sigma(t) \right) Q_{\leq M} \right\| = \mathcal{O} \left(\varepsilon^2 \sqrt{M+1} \ln(\sigma^{-1}) \right) \quad (19)$$

uniformly on bounded time intervals in $\mathcal{L}(\mathcal{F})$ and $\mathcal{L}(D(H_f))$.

Proof. Using again the Duhamel argument together with Proposition 14 we find

$$\begin{aligned} U_{\text{dress}}^\sigma(t) Q_{\leq M} &= U_0(t) Q_{\leq M} + \frac{i}{\varepsilon} U_{\text{dress}}^\sigma(t) \int_0^t ds U_{\text{dress}}^{\sigma*}(s) \varepsilon^2 \dot{\Phi}_2(t) U_0(t) Q_{\leq M} \\ &= U_0(t) Q_{\leq M} + i\varepsilon U_{\text{dress}}^\sigma(t) \int_0^t ds U_{\text{dress}}^{\sigma*}(s) Q_{\leq M+1} \dot{\Phi}_2(t) Q_{\leq M} U_0(t) \\ &= U_0(t) Q_{\leq M} + i\varepsilon U_0(t) \int_0^t ds U_0^*(s) \dot{\Phi}_2(t) Q_{\leq M} U_0(t) + \mathcal{O} \left(\varepsilon^2 \ln(\sigma^{-1}) \right) \end{aligned}$$

in the norm of $\mathcal{L}(D(H_f))$. Inspecting (18) shows that

$$\dot{\Phi}_2 = h_{\text{rad}}^\sigma + \Phi \left(\sum_{j=1}^N \frac{e_j \hat{\varphi}_\sigma(k)}{|k|^{\frac{1}{2}}} e^{-ik \cdot x_j} \langle \kappa, \dot{x}_j \rangle^2 \right) =: h_{\text{rad}}^\sigma + \Phi(f_\sigma).$$

Hence (19) follows, once we show that the second term does not contribute to the leading order transitions. This follows from an integration by parts,

$$\begin{aligned} \int_0^t ds e^{iH_f \frac{s}{\varepsilon}} \Phi(f_\sigma(s)) e^{-iH_f \frac{s}{\varepsilon}} Q_{\leq M} &= \int_0^t ds \Phi \left(e^{-i|k| \frac{s}{\varepsilon}} f_\sigma(s) \right) Q_{\leq M} \\ &= i\varepsilon \int_0^t ds \Phi \left(\frac{1}{|k|} \left(\frac{d}{ds} e^{-i|k| \frac{s}{\varepsilon}} \right) f_\sigma(s) \right) Q_{\leq M} \\ &= i\varepsilon \Phi \left(\frac{1}{|k|} e^{-i|k| \frac{s}{\varepsilon}} f_\sigma(s) \right) Q_{\leq M} \Big|_0^t - i\varepsilon \int_0^t ds \Phi \left(\frac{1}{|k|} e^{-i|k| \frac{s}{\varepsilon}} \left(\frac{d}{ds} f_\sigma(s) \right) \right) Q_{\leq M} \\ &= \mathcal{O} \left(\varepsilon \sqrt{M+1} \ln(\sigma^{-1}) \right). \quad \square \end{aligned}$$

Now Theorem 1 and 2 follow from Proposition 15 and Proposition 11. To see this put $\sigma(\varepsilon) := \varepsilon^8$ and define

$$V^\varepsilon(t) := V_{\sigma(\varepsilon)}^\varepsilon(t) \quad \text{and} \quad U_{\text{eff}}^\varepsilon(t) := U_{\text{eff}}^{\sigma(\varepsilon)}(t).$$

Then

$$\begin{aligned} \left\| \left(U_{\text{dress}}(t) - U_{\text{eff}}^\varepsilon(t) \right) Q_{\leq M} \right\| &\leq \left\| \left(U_{\text{dress}}(t) - U_{\text{dress}}^{\sigma(\varepsilon)}(t) \right) Q_{\leq M} \right\| + \left\| \left(U_{\text{dress}}^{\sigma(\varepsilon)}(t) - U_{\text{eff}}^\varepsilon(t) \right) Q_{\leq M} \right\| \\ &= \mathcal{O}(\varepsilon^7 \sqrt{M+1}) + \mathcal{O}(\varepsilon^2 \ln(\varepsilon^{-1}) \sqrt{M+1}). \end{aligned}$$

3.6 Non-adiabatic transitions

It is now evident how to define the part of the wave function that corresponds to emitted photons. As an immediate consequence of Theorem 2 we obtain the following corollary.

Corollary 16. *Let $\phi(0) \in Q_m D(H_f)$ and $\sigma(\varepsilon) = \varepsilon^8$ then*

$$\phi_{\text{su}}(t) := Q_m U_{\text{dress}}(t, 0) \phi(0) \quad \text{and} \quad \phi_{\text{na}}(t) := Q_m^\perp U_{\text{dress}}(t, 0) \phi(0)$$

have the expansions

$$\phi_{\text{su}}(t) = e^{-\frac{i}{\varepsilon}(H_f t + \int_0^t E^\varepsilon(s) ds)} \phi(0) + R_1$$

and

$$\phi_{\text{na}}(t) = i\varepsilon e^{-\frac{i}{\varepsilon}(H_f t + \int_0^t E(s) ds)} \int_0^t ds e^{iH_f \frac{s}{\varepsilon}} h_{\text{rad}}(s) e^{-iH_f \frac{s}{\varepsilon}} \phi(0) + R_2$$

with $\|R_1\|_{D(H_f)} = \mathcal{O}(\varepsilon^2 \ln(\varepsilon^{-1}))$ and $\|R_2\|_{D(H_f)} = \mathcal{O}(\varepsilon^2 \ln(\varepsilon^{-1}))$.

The probability for emitting a photon until time t is thus given by $\|\phi_{\text{na}}(t)\|^2$. But this has no simple asymptotics for $\varepsilon \rightarrow 0$ because of the infrared problem. However, for the radiated energy there is a simple asymptotic expression.

Recall the definition of the energy of free photons in (9). Note first that at any time t when $\dot{x}(t) = \ddot{x}(t) = 0$, it holds that $V^\varepsilon(t) = V_{\sigma(\varepsilon)}(t)$ and $E^\varepsilon(t) = E(t)$ and thus with Corollary 8

$$\begin{aligned} E_{\text{rad}}(\psi(t)) &= \langle V_{\sigma(\varepsilon)}(t) \psi(t), H_f V_{\sigma(\varepsilon)}(t) \psi(t) \rangle = \langle \psi(t), H^{\sigma(\varepsilon)} \psi(t) \rangle - E_{\sigma(\varepsilon)}(t) \\ &= \langle \psi(t), H \psi(t) \rangle - E(t) + \mathcal{O}(\sigma(\varepsilon)^{\frac{1}{2}}) = E_{\text{rad,stat}}(\psi(t)) + \mathcal{O}(\varepsilon^4). \end{aligned}$$

In the third equality we used equation (14) to bound $\|H - H^\sigma\|_{\mathcal{L}(D(H_f), \mathcal{F})}$ by $C\sigma^{1/2}$ and (16). This shows that whenever the energy of free photons is unambiguously defined, our definition using the superadiabatic approximation agrees with it.

We will now prove Theorem 3 by plugging the superadiabatic approximation into the definition (9). Note that this computation of the energy is the reason that we insisted on all our estimates being valid also in $\mathcal{L}(D(H_f))$.

Proof of Theorem 3. For (11) observe that with the splitting given in Corollary 16 we have that

$$\begin{aligned} E_{\text{rad}}(\psi(t)) &= \langle \phi(t), H_f \phi(t) \rangle = \langle \phi_{\text{su}}(t) + \phi_{\text{na}}(t), H_f (\phi_{\text{su}}(t) + \phi_{\text{na}}(t)) \rangle \\ &= \langle \phi_{\text{na}}(t), H_f \phi_{\text{na}}(t) \rangle = E_{\text{rad}}^0(t) + 2\text{Re}\langle R_2, H_f \phi_{\text{na}}(t) \rangle + \langle R_2, H_f R_2 \rangle, \end{aligned}$$

where $E_{\text{rad}}^0(t)$ just stands for the explicit expression claimed in (11). For the last term $\langle R_2, H_f R_2 \rangle = \mathcal{O}(\varepsilon^4 \ln(\varepsilon^{-1})^2)$ follows from $\|R_2\|_{D(H_f)} = \mathcal{O}(\varepsilon^2 \ln(\varepsilon^{-1}))$ and for the mixed term

$$|\langle R_2, H_f \phi_{\text{na}}(t) \rangle| \leq \|R_2\|_{\mathcal{F}} \|H_f \phi_{\text{na}}(t)\|_{\mathcal{F}} = \mathcal{O}(\varepsilon^4 (\ln(\varepsilon^{-1})^2))$$

from integrating $H_f \phi_{\text{na}}(t)$ by parts as in the proof of Proposition 11.

In order to get also (12), we first transform the integral (11) with $\tau' = \frac{s-s'}{\varepsilon}$, $\tau = \frac{s+s'}{2}$ and define $a(\tau) := \min(\frac{2\tau}{\varepsilon}, \frac{2}{\varepsilon}(t-\tau))$. The resulting integral is

$$I(t) := \frac{\varepsilon^3}{2} \sum_{i,j=1}^N e_j e_i \int_{S^2} d\kappa \int_0^\infty dr |\hat{\varphi}_\sigma(r)|^2 \int_0^t \int_{-a(\tau)}^{a(\tau)} d\tau d\tau' e^{ir(\tau' - \beta_{ji}(\tau, \varepsilon\tau', \kappa))} \alpha_{ji}(\tau, \varepsilon\tau', \kappa),$$

where we abbreviated $k = \kappa r$, $\alpha_{ji}(\tau, \varepsilon\tau', \kappa) := \kappa \cdot \ddot{x}_j(\tau + \frac{\varepsilon\tau'}{2}) \kappa \cdot \ddot{x}_i(\tau - \frac{\varepsilon\tau'}{2})$ and $\beta_{ji}(\tau, \varepsilon\tau', \kappa) := \kappa \cdot (x_j(\tau + \frac{\varepsilon\tau'}{2}) - x_i(\tau - \frac{\varepsilon\tau'}{2}))$. Note that α_{ji} and β_{ji} are uniformly bounded on the domain of integration. Thus we can replace $|\hat{\varphi}_\sigma(r)|^2$ by $|\hat{\varphi}(r)|^2$ with an error of order $\varepsilon^2 \sigma |t|^2$, which is negligible.

Observing that after summation the integrand is symmetric with respect to change of sign in r and τ' , i.e. for $f_{ji}(r, \tau') := |\hat{\varphi}(r)|^2 e^{ir(\tau' - \beta_{ji}(\tau, \varepsilon\tau', \kappa))} \alpha_{ji}(\tau, \varepsilon\tau', \kappa)$ it holds that $\sum_{i,j} f_{ji}(r, \tau') = \sum_{i,j} f_{ji}(-r, -\tau')$, we can extend the r -integration to all of \mathbb{R} at a cost of a factor $\frac{1}{2}$. Modulo higher order terms this leads to

$$\begin{aligned} I(t) &= \frac{\varepsilon^3}{4} \sum_{i,j=1}^N e_j e_i \int_{S^2} d\kappa \int_0^t \int_{-a(\tau)}^{a(\tau)} d\tau d\tau' \alpha_{ji}(\tau, \varepsilon\tau', \kappa) \int_{-\infty}^{\infty} dr |\hat{\varphi}(r)|^2 e^{ir(\tau' - \beta_{ji}(\tau, \varepsilon\tau', \kappa))} \\ &= \frac{\sqrt{2\pi}\varepsilon^3}{4} \sum_{i,j=1}^N e_j e_i \int_{S^2} d\kappa \int_0^t \int_{-a(\tau)}^{a(\tau)} d\tau d\tau' \alpha_{ji}(\tau, \varepsilon\tau', \kappa) (\mathcal{F}|\hat{\varphi}|^2)(\tau' - \beta_{ji}(\tau, \varepsilon\tau', \kappa)), \end{aligned}$$

where \mathcal{F} denotes the Fourier transformation. Since with φ also $(\mathcal{F}|\hat{\varphi}|^2)$ is a Schwartz function and since α_{ji} and β_{ji} are uniformly bounded on the domain of integration, we can shrink $a(\tau)$ to τ_0 with an error that is asymptotically smaller than any inverse power of τ_0 in the region where $\tau_0 < a(\tau)$. The same replacement in the region where $\tau_0 > a(\tau)$ leads to an error of order $\mathcal{O}(\varepsilon^4\tau_0)$. Taking e.g. $\tau_0 = \varepsilon^{-1/2}$ leads to negligible errors. Now we can Taylor expand the integrand as

$$\begin{aligned} e^{ir(\tau' - \beta_{ji}(\tau, \varepsilon\tau'))} \alpha_{ji}(\tau, \varepsilon\tau') &= e^{ir(\tau' - \beta_{ji}(\tau, 0) + \mathcal{O}(\varepsilon\tau_0))} (\alpha_{ji}(\tau, 0) + \mathcal{O}(\varepsilon\tau_0)) \\ &= e^{ir(\tau' - \beta_{ji}(\tau, 0))} \alpha_{ji}(\tau, 0) + \mathcal{O}((1+r)\varepsilon\tau_0) \end{aligned}$$

and the remaining leading order contribution is

$$\begin{aligned} I(t) &= \frac{\varepsilon^3}{4} \sum_{i,j=1}^N e_j e_i \int_{S^2} d\kappa \int_0^t d\tau \kappa \cdot \ddot{x}_j(\tau) \kappa \cdot \ddot{x}_i(\tau) \int_{-\tau_0}^{\tau_0} d\tau' \int_{-\infty}^{\infty} dr |\hat{\varphi}(r)|^2 e^{ir\tau'} e^{-ir\kappa \cdot (x_j(\tau) - x_i(\tau))} \\ &= \frac{\varepsilon^3}{4} \sum_{i,j=1}^N e_j e_i \int_{S^2} d\kappa \int_0^t d\tau \kappa \cdot \ddot{x}_j(\tau) \kappa \cdot \ddot{x}_i(\tau) \int_{-\tilde{\tau}_0}^{\tilde{\tau}_0} d\tilde{\tau} \int_{-\infty}^{\infty} dr |\hat{\varphi}(r)|^2 e^{ir\tilde{\tau}} \end{aligned}$$

with $\tilde{\tau} = \tau' - \kappa \cdot (x_j(\tau) - x_i(\tau))$. With the same argument as above we can replace $\tilde{\tau}_0$ by ∞ while making an negligible error and end up with

$$\begin{aligned} I(t) &= \frac{\sqrt{2\pi}\varepsilon^3}{4} \sum_{i,j=1}^N e_j e_i \int_{S^2} d\kappa \int_0^t d\tau \kappa \cdot \ddot{x}_j(\tau) \kappa \cdot \ddot{x}_i(\tau) \int_{-\infty}^{\infty} d\tilde{\tau} (\mathcal{F}|\hat{\varphi}|^2)(\tilde{\tau}) \\ &= \frac{2\pi\varepsilon^3}{4} |\hat{\varphi}(0)|^2 \sum_{i,j=1}^N e_j e_i \int_0^t d\tau \int_{S^2} d\kappa \kappa \cdot \ddot{x}_j(\tau) \kappa \cdot \ddot{x}_i(\tau) \\ &= \frac{2\pi\varepsilon^3}{4} \frac{1}{(2\pi)^3} \frac{4\pi}{3} \sum_{i,j=1}^N e_j e_i \int_0^t d\tau \ddot{x}_j(\tau) \cdot \ddot{x}_i(\tau). \quad \square \end{aligned}$$

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